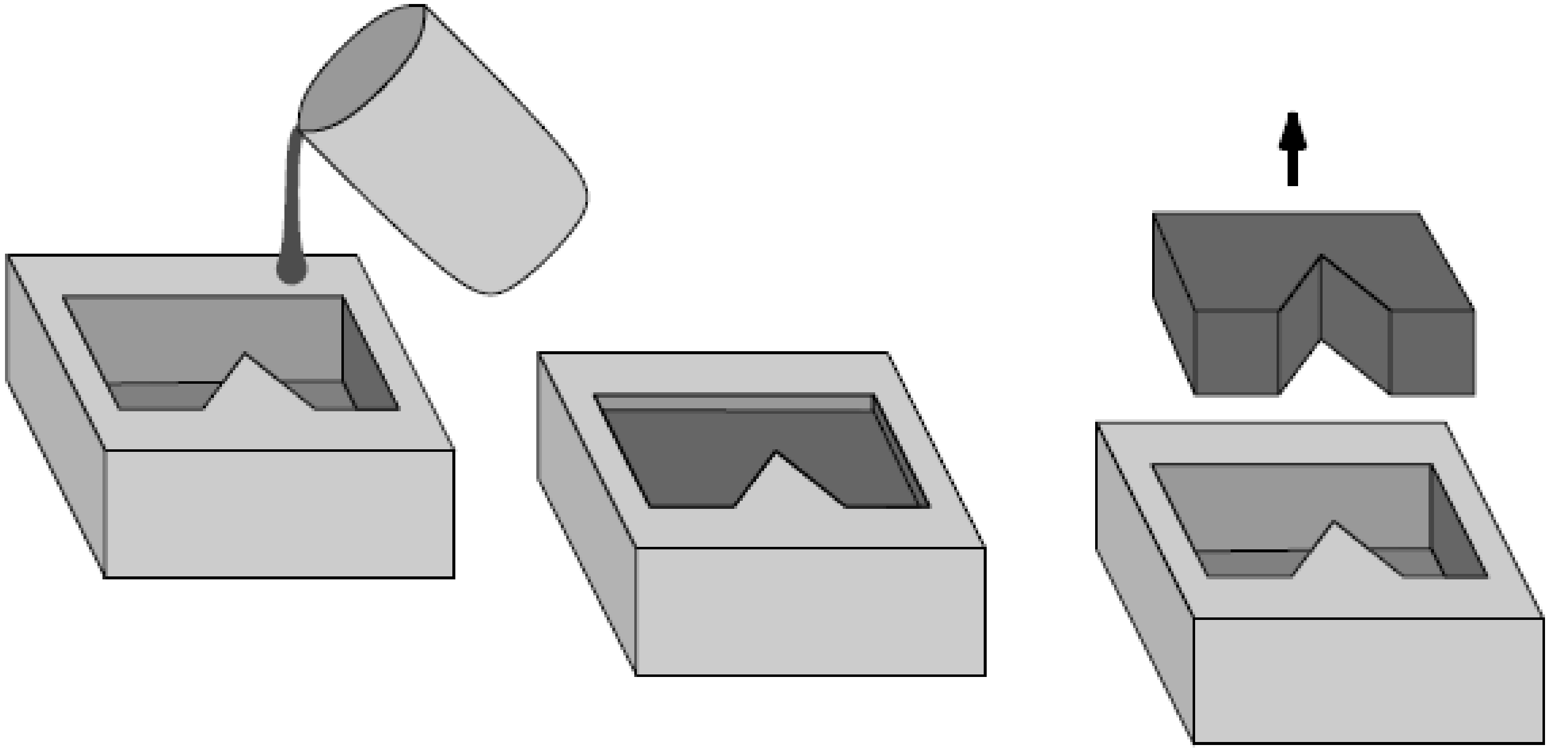
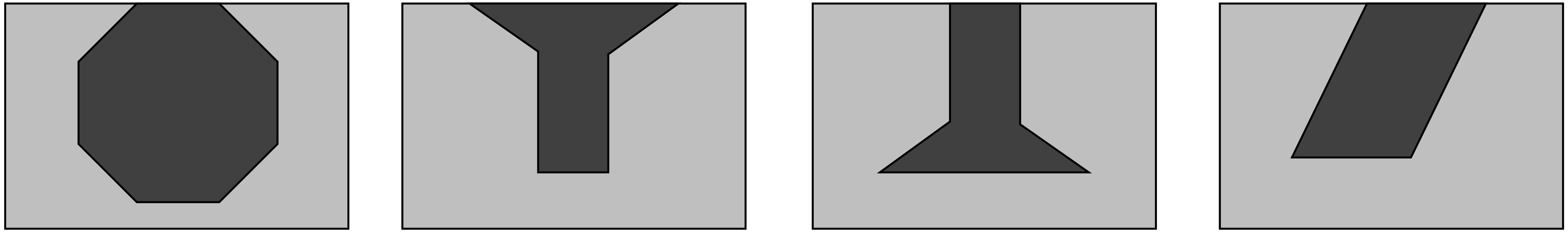
# Polyhedron Casting and Backward Analysis

# Casting



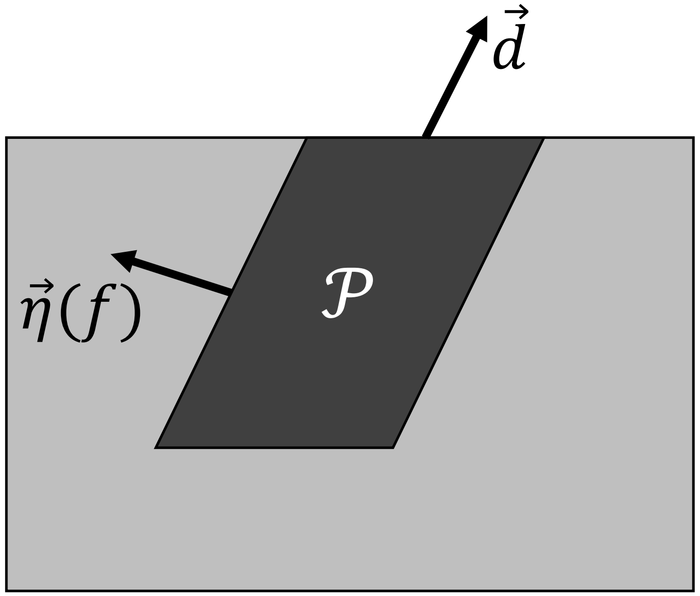
# Casting

- Can we create a mold for any polyhedron?
- If a polyhedron is castable, does any mold fit?
- Given a legal mold, in what direction we need to translate? Upwards?



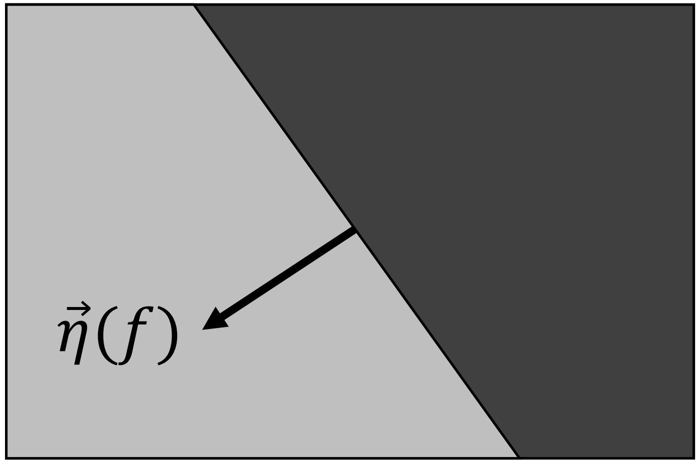
# Definitions

- A polyhedron to cast - $\mathcal{P}$
- Each face of $\mathcal{P}$, $f$, have a corresponding face in the mold $\hat{f}$.
- The (outward) normal of $f$ - $\vec{\eta}(f)$.
- Direction of translation - $\vec{d}$.



# Castable polyhedrons

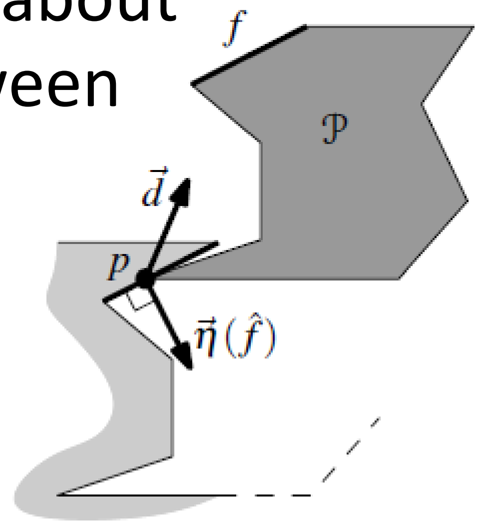
- Which derections $\vec{d}$ are valid?




- We want the angle between $\vec{d}$ and $\vec{\eta}(f)$ to be at least $90°$.
- For each face!

# Castable polyhedrons

- <u>Lemma:</u> The polyhedron $\mathcal{P}$ can be removed from its mold by a translation in direction $\vec{d}$ if and only if $\vec{d}$ makes an angle of at least $90°$ with $\vec{\eta}(f)$ for all $f$.
- Only if – We have already seen.
- If – The same reasoning holds for any collision, if $\mathcal{P}$ is about to collide with the mold at face $\hat{f}$ then the angle between $\vec{d}$ and $\vec{\eta}(f)$ is less than $90°$.



# Castable polyhedrons

- Let us write the directions as vectors –

$$\vec{d} = (d_x, d_y, 1)$$

Why 1?

$$\vec{\eta} = (\eta_x, \eta_y, \eta_z)$$

- Recall that $\vec{d} \cdot \vec{\eta} = \left|\vec{d}\right| \cdot |\vec{\eta}| \cdot \cos(\theta)$
- We want $\theta$ to be greater than $90°$ for all faces, that is:

$$\vec{d} \cdot \vec{\eta} \leq 0 \Rightarrow$$
$$d_x \eta_x + d_y \eta_y + \eta_z \leq 0$$

- How do we solve it for all faces?

# Castable polyhedrons

- We want $\theta$ to be greater than $90°$ for all faces, that is:

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- How do we solve it for all faces?
- Linear programing!
- Corollary: we can decide if a polyhedron is castable in $O(n^2)$ expected time.
  - Why $O(n^2)$?

# Backward analysis

- What is the worst case time complexity of the following algorithm?
- And the expected time?

**Algorithm** PARANOIDMAXIMUM($A$)

**if** $\text{card}(A) = 1$
  **then return** the unique element $x \in A$
  **else** Pick a random element $x$ from $A$.
    $x' \leftarrow$ PARANOIDMAXIMUM($A \setminus \{x\}$)
    **if** $x \leqslant x'$
      **then return** $x'$
      **else** Now we suspect that $x$ is the maximum, but to be absolutely sure, we compare $x$ with all $\text{card}(A) - 1$ other elements of $A$.
        **return** $x$

# Backward analysis

- Worst case:
- $T(n) = T(n-1) + O(n) = O(n^2)$

**Algorithm** PARANOIDMAXIMUM($A$)

```
if card(A) = 1
   then return the unique element x ∈ A
   else  Pick a random element x from A.
         x' ← PARANOIDMAXIMUM(A \ {x})
         if x ⩽ x'
            then return x'
            else  Now we suspect that x is the maximum, but to be
                  absolutely sure, we compare x with all card(A) − 1
                  other elements of A.
                  return x
```

# Backward analysis

- Expected time:
- $\mathrm{E}\big(T(n)\big) = \mathrm{E}\big(T(n-1)\big) + \mathrm{E}(f(n))$

$$= \mathrm{E}\big(T(n-2)\big) + \mathrm{E}\big(f(n)\big) + E\big(f(n-1)\big)$$

$$\ldots = \sum_{i=1}^{n} f(i)$$

$$= \frac{i-1}{i} O(1) + \frac{1}{i} O(i)$$

$$= O(n)$$

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